Dynamic Consistency of Conditional Simple Temporal Networks via Mean Payoff Games

A Singly-Exponential Time DC-Checking

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22nd International Symposium on
Temporal Representation and Reasoning
University of Kassel, Kassel, Germany
September 23, 2015
Simple Temporal Networks (STNs) [Dechter, Meiri, Pearl [3]]

- Represent a general framework for analyzing systems (conjunctions) of difference constraints on ordered pairs of temporal variables.
- An STN can be encoded by a weighted directed graph:
  - a node represents a time-point variable (time-point);
  - an arc represents a temporal distance constraint:
    - $u \rightarrow^a v$ stands for $v - u \leq a$, $a \in \mathbb{R}$;
    - $(v \leftarrow^a u$ stands for $v - u \geq a$).

Represented Constraints:

- $4 \leq A - D \leq 5$
- $6 \leq B - A \leq 7$
- $1 \leq C - A \leq 4$
- $-1 \leq B - C \leq 1$
- $1 \leq C - D \leq 3$
STN Consistency [Dechter, Meiri, Pearl [3]]

An STN $\langle G = (V, E), \ell \rangle$ is consistent if it admits a feasible scheduling function, i.e., we can assign a real value $s(v)$ to each time-point $v$, such that all constraints are satisfied:

$$\exists \ s : V \mapsto \mathbb{R} \ \text{such that:}$$

$$s(v) \leq s(u) + \ell(u,v) \quad \forall (u, v) \in E.$$

An STN is not consistent if it contains a negative cycle.

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A Singly-Exponential Time DC-Checking of CSTNs via MPGs
C. Comin and R. Rizzi
The CSTN formalism extends STNs in that:

1. some of the nodes are called observation events and to each of them is associated a boolean variable, to be disclosed only at execution time;
2. labels (i.e. conjunctions over the literals) are attached to all nodes and constraints, to indicate the situations in which each of them is required.
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Conditional Simple Temporal Networks: the Model


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Let \( s(p) = \top \) and \( s(q) = \bot \)...

![Diagram](https://via.placeholder.com/150)
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$$s : P \rightarrow \{\top, \bot\}.$$

Let $s(p) = \top$ and $s(q) = \bot$, then $\Gamma$ becomes:

![Diagram](image-url)
The restriction $\Gamma^+_s$. 

- The **restriction** of $V$ and $A$ w.r.t. the scenario $s \in \Sigma_P$ are:
  - $V^+_s \triangleq \{ v \in V \mid s(L(v)) = T \}$;
  - $A^+_s \triangleq \{ \langle u, v, w \rangle \mid \exists \ell \langle v - u \leq w, \ell \rangle \in A, s(\ell) = T \}$.

- The **restriction** of $\Gamma$ w.r.t. $s$ is the STN $\Gamma^+_s \triangleq \langle V^+_s, A^+_s \rangle$. 

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![Diagram of Conditional Simple Temporal Networks]
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![Diagram with conditional edges and labels]
CSTNs: Consistency

CSTN’s Consistencies

Three notions of consistency arise for CSTNs: *weak*, *strong*, and...

**dynamic consistency.**
CSTNs: Consistency

**CSTN’s Dynamic Consistency**

*Dynamic Consistency (DC)* requires the existence of conditional plans where *decisions* about precise timing of actions are *postponed until exec. time*, but it guarantees that all the relevant constraints will be ultimately satisfied.
CSTNs: Dynamic Consistency

An example of Dynamic execution for the CSTN $\Gamma$.

Let $\phi(A) = 0$, 

\[
\begin{align*}
\phi(A) &= 0, \\
A &\rightarrow B, \\
B &\rightarrow C, \\
C &\rightarrow A.
\end{align*}
\]
CSTNs: Dynamic Consistency

An example of Dynamic execution for the CSTN $\Gamma$.

Let $\phi(A) = 0$, $\phi(O_p) = 1$, 

\[ \begin{align*}
A & \rightarrow 0, \quad 3, p \land \neg q \\
B & \rightarrow 10, \quad -10 \\
C & \rightarrow 2, q \\
O_p & \rightarrow 5, 0 \\
O_q & \rightarrow 9, 0
\end{align*} \]

1, $\neg p$
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An example of Dynamic execution for the CSTN $\Gamma$.

Let $\phi(A) = 0$, $\phi(O_p) = 1$, $s(p) = \top$, $\phi(O_q) = 2$, 

\[
\begin{align*}
A & \xrightarrow{0} B \\
& \xvdash 3, \neg q \\
& \xrightarrow{5} 0 \\
& \xrightarrow{9} 0 \\
C & \xleftarrow{10} A \\
& \xrightarrow{10} 2, q \\
& \xleftarrow{2} 2 \\
& \xleftarrow{q?} O_q
\end{align*}
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CSTNs: Dynamic Consistency

An example of Dynamic execution for the CSTN $\Gamma$.

Let $\phi(A) = 0$, $\phi(O_p) = 1$, $s(p) = \top$, $\phi(O_q) = 2$, $s(q) = \bot$, $\phi(B) = 3$, $\phi(C') = 10$. 
An **Execution Strategy** for $\Gamma$ is a mapping $\sigma : \Sigma_P \rightarrow \Phi_V$ such that, for any scenario $s \in \Sigma_P$, the domain of the scheduling $\sigma(s) \in \Phi_V$ is $V_s^+$. 

\[
\begin{align*}
\phi(A) &= 0 \\
\phi(O_p) &= 1 \\
\phi(O_q) &= 2 \\
\phi(B) &= 8 \\
\phi(C) &= 10 \\
\phi(O_q) &= 9 \\
\phi(C) &= 10
\end{align*}
\]
We say that $\sigma \in S_\Gamma$ is a **viable** execution strategy if, for each scenario $s \in \Sigma_P$, the scheduling $\sigma(s) \in \Phi_V$ is **feasible** for the STN $\Gamma_{s+}^\Gamma$. 

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CSTNs: Dynamic Consistency

Difference Set $\Delta(s_1; s_2)$

Let $s_1, s_2 \in \Sigma_P$ be two scenarios. The set of observation events in $V_{s_1}^+ \cap O V$ at which $s_1$ and $s_2$ differ is denoted by $\Delta(s_1; s_2)$.
Difference Set $\Delta(s_1; s_2)$

Formally,

$$\Delta(s_1; s_2) \triangleq \{ O_p \in V_{s_1}^+ \cap O V \mid s_1(p) \neq s_2(p) \}.$$
Let $\sigma \in S_\Gamma$ be an execution strategy. Let $s \in \Sigma_P$ be a scenario. The **scheduling time** of $u \in V$ in $\sigma(s)$ is denoted $[\sigma(s)]_u = \phi(u)$ for fixed $s, \sigma$.
Let $\sigma \in S_T$ be an execution strategy. Then, $\sigma$ is **dynamic** if and only if the following implication holds for every $s_1, s_2 \in \Sigma_P$, $u \in V_{s_1,s_2}^+$:

$$\left( \bigwedge_{v \in \Delta(s_1;s_2)} [\sigma(s_1)]_u \leq [\sigma(s_1)]_v \right) \Rightarrow [\sigma(s_1)]_u = [\sigma(s_2)]_u$$
Dynamic Consistency of CSTNs: Main Facts

- A CSTN $\Gamma$ is **dynamically-consistent** if it admits a **viable and dynamic** execution strategy.
- DC-Checking was conjectured to be hard to assess [Tsamardinos, Vidal and Pollack, 2003]
- The best-so-far complexity upper-bound for the DC-Checking of CSTNs is **doubly-exponential time**.
  - Build an equivalent *Disjunctive Temporal Problem (DTP)* of size exponential in the input CSTN.
  - Apply to it an exponential time DTP’s algorithm to check its consistency.
    - Checking general DTPs is NP-complete. [Stergiou, Koubarakis, 2000]
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The questions we faced:

- **What is the computational complexity of Checking Dynamic-Consistency in CSTNs?**
  - Lower-Bound: n/a
  - Upper-Bound: 2-EXP [Tsamardinos, Vidal, Pollack, 2003]

- **Does there exist faster algorithms that can be used in practice?**
  - ... say, $|P| \sim 20$ and $|V| \sim 1000$;
  - or $|P| \sim 25$ and $|V| \sim 100$. 
Main Results

Our Contribution:
- Lower-Bound: \( \text{coNP-hard} \).
- Upper-Bound: \( \text{NE} \cap \text{coNE} \cap \text{pseudo-E} \).

A (pseudo) Singly-Exponential Time DC-Checking for CSTNs.
- There exists an
  \[
  O(|\Sigma_P|^3|A|^2|V| + |\Sigma_P|^4|A||V|^2|P| + |\Sigma_P|^5|V|^3|P|) W
  \]
  \(\text{(pseudo) singly-exponential}\) time algorithm for checking DC on any input CSTN \( \Gamma = \langle V, A, L, O, O V, P \rangle \).
  - Here, \( W \triangleq \max_{a \in A} |w_a| \) and \( |\Sigma_P| \leq 2^{|P|} \).
- In particular, given any dynamically-consistent CSTN \( \Gamma \), the algorithm returns a viable and dynamic execution strategy.

(here above, \( \text{E} \) is deterministic singly-exponential time, \( \text{NE} \) is nondeterministic singly-exponential time)
Main Results

Checking DC of CSTNs via MPGs

Most importantly, we unveil a connection between the problem of checking **DC in CSTNs** and that of determining **Mean Payoff Games**.

- Recently, STNs have been generalized into **Hyper Temporal Networks** (HyTNs) [1, 2] by considering weighted directed hypergraphs, where each hyperarc models a *disjunctive* temporal constraint called **hyper-constraint**.
- The computational equivalence between checking the consistency of HyTNs and determining winning regions in (MPGs) was pointed out.
- The present work unveils that HyTNs and MPGs are a suitable underlying combinatorial model for the DC-Checking of CSTNs.
Main Results

ε-Dynamic Consistency and the Reaction Time $\hat{\epsilon}(\Gamma)$

- In order to analyze the algorithm, we introduce a novel and refined notion of dynamic-consistency, named $\epsilon$-dynamic-consistency;
  - We provide a sharp lower bounding analysis of the critical value of the reaction time $\hat{\epsilon}(\Gamma)$ where the CSTN $\Gamma$ transits from being, to not being, dynamically-consistent.
  - This clarifies the role of the reaction time $\hat{\epsilon}$ in the DC-checking of CSTNs.
Let us provide a sketch of the arguments...
Checking DC of CSTNs is coNP-hard

Sketch of coNP-hardness proof

We reduce 3-SAT to the complement of CSTN-DC.

\( \varphi(x_1, \ldots, x_n) = \bigwedge_{i=1}^{m} (\alpha_i \lor \beta_i \lor \gamma_i) \) is mapped to the following CSTN:
Let us recall Hyper Temporal Networks...
A more general arc constraint

A hypergraph $\mathcal{H}$ is a pair $(V, \mathcal{A})$, where $V$ is the set of nodes, and $\mathcal{A}$ is the set of hyperarcs. Each hyperarc $A \in \mathcal{A}$ has a distinguished node $t_A$, called the tail of $A$, and a nonempty weighted set $(H_A, w_A)$, where $H_A \subseteq V \setminus \{t_A\}$ contains the heads of $A$, and each head $v \in H_A$ is associated with a weight $w_A(v) \in \mathbb{R}$. 
An HyTN $\mathcal{H} = (V, A)$ is consistent if it admits a feasible scheduling function, i.e., we can assign a real value $\phi(v)$ to each time-point $v$, such that each hyperarc constraint is satisfied:

$$\phi(t_A) \geq \min_{v \in H_A} \{\phi(v) - w_A(v)\} \quad \forall A \in A.$$ (1)
Theorem

The following propositions hold on HyTNs.

1. There exists an $O((|V| + |A|) m_A W)$ pseudo-polynomial time algorithm for checking HyTN-Consistency;

2. There exists an $O((|V| + |A|) m_A W)$ pseudo-polynomial time algorithm such that, given in input any consistent HyTN $\mathcal{H} = (V, A)$, it returns as output a feasible scheduling $\phi : V \rightarrow \mathbb{R}$ of $\mathcal{H}$;

Here, $W \triangleq \max_{A \in A, v \in H_A} |w_A(v)|$.

The approach was shown to be robust by experimental evaluations [Comin, Posenato, Rizzi [2]], where (randomly generated) HyTNs of size up to $|V| \sim 10^6$ and $W \sim 10^3$ were solved.
Experimental Evaluation of the **HyTN** Algorithm

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\mu$ (sec)</th>
<th>$\sigma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1 \cdot 10^5$</td>
<td>0.83</td>
<td>8.64</td>
</tr>
<tr>
<td>$2 \cdot 10^5$</td>
<td>1.18</td>
<td>6.55</td>
</tr>
<tr>
<td>$3 \cdot 10^5$</td>
<td>1.80</td>
<td>9.46</td>
</tr>
<tr>
<td>$4 \cdot 10^5$</td>
<td>2.19</td>
<td>9.14</td>
</tr>
<tr>
<td>$5 \cdot 10^5$</td>
<td>2.42</td>
<td>6.06</td>
</tr>
<tr>
<td>$6 \cdot 10^5$</td>
<td>3.37</td>
<td>12.80</td>
</tr>
<tr>
<td>$7 \cdot 10^5$</td>
<td>3.68</td>
<td>8.77</td>
</tr>
<tr>
<td>$8 \cdot 10^5$</td>
<td>3.53</td>
<td>6.16</td>
</tr>
<tr>
<td>$9 \cdot 10^5$</td>
<td>4.24</td>
<td>7.95</td>
</tr>
<tr>
<td>$10 \cdot 10^5$</td>
<td>4.54</td>
<td>8.38</td>
</tr>
</tbody>
</table>

(a) Test 1 results.

(b) Graphical representation of the results.

**Test1’s Data Set:** 1000 HyTN per size, $\forall v \in V \ deg(v) = 3$, $W = 1000$. 

A Singly-Exponential Time DC-Checking of CSTNs via MPGs

C. Comin and R. Rizzi
Sketch of the reduction from CSTN-Dynamic-Consistency to HyTN-Consistency

Sketch of the reduction

- Introduce $\epsilon$-dynamic-consistency.
- Prove that any execution strategy $\sigma$ is dynamic iff $\sigma$ is $\epsilon$-dynamic for some real number $\epsilon \in (0, +\infty)$.
- Consider $\hat{\epsilon}(\Gamma) = \sup\{\epsilon \in \mathbb{R}_{>0} \mid \Gamma$ is $\epsilon$-dynamically-consistent\}.
  - $\hat{\epsilon}(\Gamma)$ is the reaction time of $\Gamma$.
- Prove that for any dynamically-consistent CSTN $\Gamma$, where $V$ is the set of events and $\Sigma_P$ is the set of scenarios, it holds $\hat{\epsilon}(\Gamma) \geq |\Sigma_P|^{-1}|V|^{-1}$.
- Devise an algorithm for checking $\epsilon$-dynamic-consistency by reducing that problem to the consistency checking of HyTNs.
- (Bonus) Prove that the bound $\hat{\epsilon}(\Gamma) \geq |\Sigma_P|^{-1}|V|^{-1}$ is (almost) optimal.
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\( \varepsilon \)-dynamic-consistency

Given any CSTN \( \langle V, A, L, O, O^V, P \rangle \) and any real number \( \varepsilon \in (0, +\infty) \), an execution strategy \( \sigma \in S_\Gamma \) is \( \varepsilon \)-dynamic if it satisfies all the \( H_\varepsilon \)-constraints, namely, for any two scenarios \( s_1, s_2 \in \Sigma_P \) and any event \( u \in V^+_{s_1,s_2} \), the execution strategy \( \sigma \) satisfies the following constraint, denoted \( H_\varepsilon(s_1; s_2; u) \):

\[
[\sigma(s_1)]_u \geq \min \left( \{[\sigma(s_2)]_u\} \cup \{[\sigma(s_1)]_v + \varepsilon \mid v \in \Delta(s_1; s_2)\} \right)
\]

We say that a CSTN \( \Gamma \) is \( \varepsilon \)-dynamically-consistent if it admits \( \sigma \in S_\Gamma \) which is both viable and \( \varepsilon \)-dynamic.
Lemma 1

If $\Gamma$ is $\epsilon$-dynamically-consistent, for some $\epsilon > 0$, then $\Gamma$ is $\epsilon'$-dynamically-consistent for every $\epsilon' \in (0, \epsilon]$. 

$\epsilon$-dynamic-consistency
Lemma 2

Let $\sigma$ be a dynamic execution strategy for the CSTN $\Gamma$. Then, there exists a sufficiently small real number $\epsilon \in (0, +\infty)$ such that $\sigma$ is $\epsilon$-dynamic.
Lemma 3

Let $\sigma$ be an $\epsilon$-dynamic execution strategy for the CSTN $\Gamma$, for some $\epsilon \in (0, +\infty)$. Then, $\sigma$ is dynamic.

$\Rightarrow$ Dynamic-Consistency of CSTNs is expressible with Max-Plus (or Min-Plus) constraints.
Solving $\epsilon$-dynamic-consistency: Expansion of a CSTN

### Expansion $\langle V^\text{Ex}_\Gamma, \Lambda^\text{Ex}_\Gamma \rangle$

Let $\Gamma$ be a CSTN $\langle V, A, L, O, OV, P \rangle$. Consider the distinct STNs $\langle V_s, A_s \rangle$, one for each scenario $s \in \Sigma_P$, defined as follows:

$$V_s \triangleq \{ v_s \mid v \in V_s^+ \} \quad \text{and} \quad A_s \triangleq \{ \langle u_s, v_s, w \rangle \mid \langle u, v, w \rangle \in A_s^+ \}.$$

We define the expansion $\langle V^\text{Ex}_\Gamma, \Lambda^\text{Ex}_\Gamma \rangle$ of $\Gamma$ as follows:

$$\langle V^\text{Ex}_\Gamma, \Lambda^\text{Ex}_\Gamma \rangle \triangleq \left( \bigcup_{s \in \Sigma_P} V_s, \bigcup_{s \in \Sigma_P} A_s \right).$$
Solving $\epsilon$-dynamic-consistency: Expansion of a CSTN

A Singly-Exponential Time DC-Checking of CSTNs via MPGs

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Solving $\epsilon$-dynamic-consistency: Expansion of a CSTN

An excerpt of the expansion of the CSTN $\Gamma$ with two scenarios $s_1$ and $s_4$. 
Given any $\epsilon \in (0, +\infty)$ and any CSTN $\Gamma = \langle V, A, L, O, O^V, P \rangle$, a corresponding HyTN denoted by $H_\epsilon(\Gamma)$ can be defined as follows:

- For every scenarios $s_1, s_2 \in \Sigma_P$ and every event $u \in V^+_{s_1, s_2}$, define a hyperarc $\alpha = \alpha_\epsilon(s_1; s_2; u)$ as $\alpha_\epsilon(s_1; s_2; u) \triangleq \langle t_\alpha, H_\alpha, w_\alpha \rangle$, where:
  - $t_\alpha \triangleq u_{s_1}$ is the tail of the hyperarc $\alpha$;
  - $H_\alpha \triangleq \{ u_{s_2} \} \cup \Delta(s_1; s_2)$ is the set of the heads;
  - $w_\alpha(u_{s_2}) \triangleq 0$; $w_\alpha(v) \triangleq -\epsilon$ for each $v \in \Delta(s_1; s_2)$.

- Consider the expansion $\langle V^\text{Ex}_\Gamma, \Lambda^\text{Ex}_\Gamma \rangle$ of $\Gamma$. Then, $H_\epsilon(\Gamma)$ is defined as $H_\epsilon(\Gamma) \triangleq \langle V^\text{Ex}_\Gamma, A_{H_\epsilon} \rangle$, where:
  $$A_{H_\epsilon} \triangleq \Lambda^\text{Ex}_\Gamma \cup \bigcup_{s_1, s_2 \in \Sigma_P} \alpha_\epsilon(s_1; s_2; u).$$
Solving $\epsilon$-DC: reducing from CSTN $\Gamma$ to HyTN $H_\epsilon(\Gamma)$

**HyTN $H_\epsilon(\Gamma)$**

Given any $\epsilon \in (0, +\infty)$ and any CSTN $\Gamma = \langle V, A, L, O, O^V, P \rangle$, a corresponding HyTN denoted by $H_\epsilon(\Gamma)$ can be defined as follows:

- For every scenarios $s_1, s_2 \in \Sigma_P$ and every event $u \in V_{s_1,s_2}^+$, define a hyperarc $\alpha = \alpha_\epsilon(s_1; s_2; u)$ as $\alpha_\epsilon(s_1; s_2; u) \triangleq \langle t_\alpha, H_\alpha, w_\alpha \rangle$, where:
  - $t_\alpha \triangleq u_{s_1}$ is the tail of the hyperarc $\alpha$;
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- Consider the expansion $\langle V^{Ex}_\Gamma, \Lambda^{Ex}_\Gamma \rangle$ of $\Gamma$.

Then, $H_\epsilon(\Gamma)$ is defined as $H_\epsilon(\Gamma) \triangleq \langle V^{Ex}_\Gamma, A_{H_\epsilon} \rangle$, where:

$$A_{H_\epsilon} \triangleq \Lambda^{Ex}_\Gamma \cup \bigcup_{s_1,s_2 \in \Sigma_P} \alpha_\epsilon(s_1; s_2; u).$$
Given any $\epsilon \in (0, +\infty)$ and any CSTN $\Gamma = \langle V, A, L, O, O V, P \rangle$, a corresponding HyTN denoted by $\mathcal{H}_\epsilon(\Gamma)$ can be defined as follows:

- For every scenarios $s_1, s_2 \in \Sigma_P$ and every event $u \in V^+_{s_1, s_2}$, define a hyperarc $\alpha = \alpha_\epsilon(s_1; s_2; u)$ as $\alpha_\epsilon(s_1; s_2; u) \triangleq \langle t_\alpha, H_\alpha, w_\alpha \rangle$, where:
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  - $w_\alpha(u_{s_2}) \triangleq 0$; $w_\alpha(v) \triangleq -\epsilon$ for each $v \in \Delta(s_1; s_2)$.

- Consider the expansion $\langle V^{\text{Ex}}_\Gamma, \Lambda^{\text{Ex}}_\Gamma \rangle$ of $\Gamma$.
  Then, $\mathcal{H}_\epsilon(\Gamma)$ is defined as $\mathcal{H}_\epsilon(\Gamma) \triangleq \langle V^{\text{Ex}}_\Gamma, \mathcal{A}_{\mathcal{H}_\epsilon} \rangle$, where:

\[
\mathcal{A}_{\mathcal{H}_\epsilon} \triangleq \Lambda^{\text{Ex}}_\Gamma \cup \bigcup_{s_1, s_2 \in \Sigma_P} \bigcup_{u \in V^+_{s_1, s_2}} \alpha_\epsilon(s_1; s_2; u).
\]
Solving $\epsilon$-DC: reducing from CSTN $\Gamma$ to HyTN $H_\epsilon(\Gamma)$

**HyTN $H_\epsilon(\Gamma)$**

Given any $\epsilon \in (0, +\infty)$ and any CSTN $\Gamma = \langle V, A, L, O, O V, P \rangle$, a corresponding HyTN denoted by $H_\epsilon(\Gamma)$ can be defined as follows:

- For every scenarios $s_1, s_2 \in \Sigma_P$ and every event $u \in V_{s_1,s_2}^+$, define a hyperarc $\alpha = \alpha_\epsilon(s_1; s_2; u)$ as $\alpha_\epsilon(s_1; s_2; u) \triangleq \langle t_\alpha, H_\alpha, w_\alpha \rangle$, where:
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  - $w_\alpha(u_{s_2}) \triangleq 0$; $w_\alpha(v) \triangleq -\epsilon$ for each $v \in \Delta(s_1; s_2)$.

- Consider the expansion $\langle V_{\Gamma}^{Ex}, \Lambda_{\Gamma}^{Ex} \rangle$ of $\Gamma$.
  Then, $H_\epsilon(\Gamma)$ is defined as $H_\epsilon(\Gamma) \triangleq \langle V_{\Gamma}^{Ex}, A_{H_\epsilon} \rangle$, where:

$$A_{H_\epsilon} \triangleq \Lambda_{\Gamma}^{Ex} \cup \bigcup_{s_1, s_2 \in \Sigma_P, u \in V_{s_1, s_2}^+} \alpha_\epsilon(s_1; s_2; u).$$
Solving $\epsilon$-DC: reducing from CSTN $\Gamma$ to HyTN $\mathcal{H}_\epsilon(\Gamma)$

**HyTN $\mathcal{H}_\epsilon(\Gamma)$**

Given any $\epsilon \in (0, +\infty)$ and any CSTN $\Gamma = \langle V, A, L, O, O V, P \rangle$, a corresponding HyTN denoted by $\mathcal{H}_\epsilon(\Gamma)$ can be defined as follows:

- For every scenarios $s_1, s_2 \in \Sigma_P$ and every event $u \in V_{s_1, s_2}$, define a hyperarc $\alpha = \alpha_\epsilon(s_1; s_2; u)$ as $\alpha_\epsilon(s_1; s_2; u) \triangleq \langle t_\alpha, H_\alpha, w_\alpha \rangle$, where:
  - $t_\alpha \triangleq u_{s_1}$ is the tail of the hyperarc $\alpha$;
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  - $w_\alpha(u_{s_2}) \triangleq 0$; $w_\alpha(v) \triangleq -\epsilon$ for each $v \in \Delta(s_1; s_2)$.

- Consider the expansion $\langle V_{\Gamma}^{\text{Ex}}, \Lambda_{\Gamma}^{\text{Ex}} \rangle$ of $\Gamma$. Then, $\mathcal{H}_\epsilon(\Gamma)$ is defined as $\mathcal{H}_\epsilon(\Gamma) \triangleq \langle V_{\Gamma}^{\text{Ex}}, A_{\mathcal{H}_\epsilon} \rangle$, where:

$$A_{\mathcal{H}_\epsilon} \triangleq \Lambda_{\Gamma}^{\text{Ex}} \cup \bigcup_{s_1, s_2 \in \Sigma_P} \alpha_\epsilon(s_1; s_2; u).$$
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Theorem (CSTNs and HyTNs)

Given any CSTN $\Gamma = \langle V, A, L, O, O V, P \rangle$, there exists a sufficiently small real number $\epsilon \in (0, +\infty)$ such that $\Gamma$ is dynamically-consistent if and only if $H_\epsilon(\Gamma)$ is consistent.

Moreover, $H_\epsilon(\Gamma)$ has at most $|V_{H_\epsilon}| \leq |\Sigma_P| |V|$ nodes,

$|A_{H_\epsilon}| = O(|\Sigma_P| |A| + |\Sigma_P|^2 |V|)$ hyperarcs, and it has size at most $m_{A_{H_\epsilon}} = O(|\Sigma_P| |A| + |\Sigma_P|^2 |V| |P|)$. 

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Pseudocode of the Algorithm

Algorithm 1: check_CSTN-ε-DC(Γ = ⟨V, A, L, O, OV, P⟩, ε = N / D)

1: \( \mathcal{H}_\varepsilon(\Gamma) \leftarrow \text{construct}_\mathcal{H}(\Gamma, \varepsilon); \)
2: \textbf{foreach} \((A = \langle t_A, H_A, w_A \rangle \in \mathcal{A}_{\mathcal{H}_\varepsilon(\Gamma)} \text{ AND } h \in H_A)\) \textbf{do}
3: \quad \(w_A(h) \leftarrow D \cdot w_A(h); \) // scale weights to \( Z \)
4: \textbf{end}
5: \( \phi \leftarrow \text{check_HyTN-consistency}(\mathcal{H}_\varepsilon(\Gamma)); \)
6: \textbf{if} \( (\phi \text{ is a feasible scheduling of } \mathcal{H}_\varepsilon(\Gamma)) \) \textbf{then}
7: \quad \textbf{foreach} \((\text{event node } v \in V_{\mathcal{H}_\varepsilon(\Gamma)})\) \textbf{do}
8: \quad \quad \( \phi(v) \leftarrow \phi(v) / D; \) // re-scale back to size w.r.t \( \varepsilon \)
9: \quad \textbf{end}
10: \textbf{return} \langle \text{YES}, \phi \rangle;
11: \textbf{end}
12: \textbf{else}
13: \quad \textbf{return} \textit{NO};
14: \textbf{end}
Theorem

Let $\Gamma = \langle V, A, L, O, O V, P \rangle$ be a CSTN. Let $\epsilon \triangleq |\Sigma_P|^{-1}|V|^{-1}$.

Then, $\Gamma$ is dynamically-consistent if and only if $\Gamma$ is $\epsilon$-dynamically-consistent.

**Algorithm 2: check\_DC($\Gamma = \langle V, A, L, O, O V, P \rangle$)**

1. $\hat{\epsilon} \leftarrow |\Sigma_P|^{-1}|V|^{-1}$;
2. return \texttt{check\_CSTN\textparentheses{$\epsilon$}-DC($\Gamma$, $\hat{\epsilon}$)};
Remark

Hyperarc constraints can also be allowed inside the input CSTNs, besides the standard arc constraints.

Thus, the algorithm actually solves a larger family of conditional temporal networks, that one may call: Conditional Hyper Temporal Networks (CHyTNs).
The bound $\hat{\epsilon}(\Gamma) \geq |\Sigma_P|^{-1} |V|^{-1}$ is sharp.

- A natural question is whether the lower bound $\hat{\epsilon}(\Gamma) \geq |\Sigma_P|^{-1} |V|^{-1}$ can be improved up to $\hat{\epsilon}(\Gamma) = \Omega(|V|^{-1})$...
- ... this would improve the time complexity by a factor $|\Sigma_P|$.
- However, the following theorem shows that this is not the case by exhibiting a CSTN for which $\hat{\epsilon}(\Gamma) = 2^{-\Omega(|P|)}$.
- This proves that the bound $\hat{\epsilon}(\Gamma) \geq |\Sigma_P|^{-1} |V|^{-1}$ is (almost) sharp.

**Theorem**

For each $n \in \mathbb{N}_0$ there exists a CSTN $\Gamma^n$ such that

$$\hat{\epsilon}(\Gamma^n) < 2^{-n+1} = 2^{-|P^n|/3+1},$$

where $P^n$ is the set of boolean variables of $\Gamma^n$. 
The bound $\hat{\epsilon}(\Gamma) \geq |\Sigma P|^{-1} |V|^{-1}$ is sharp.

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- However, the following theorem shows that this is not the case by exhibiting a CSTN for which $\hat{\epsilon}(\Gamma) = 2^{-\Omega(|P|)}$.
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**Theorem**

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How $\Gamma^n$ looks like
How $\Gamma^n$ looks like

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How $\Gamma^n$ looks like
Conclusion

Future Works and Open Problems

- Is it possible to extend the HyTN/MPG approach to check the dynamic-controllability of CSTNs with Uncertainty?
- Conduct an in-depth experimental evaluation.
- Is the DC-checking of CSTNs (decision problem) in PSPACE?
- Is it PSPACE-hard? Is it complete for some natural complexity class?
Thank you.

Thank you for your attention.


